Pricing annuity guarantees under a double regime-switching model

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This paper is concerned with the valuation of equity-linked annuities with mortality risk under a double regime-switching model, which provides a way to endogenously determine the regime-switching risk. The model parameters and the reference investment fund price level are modulated by a continuous-time, finite-time, observable Markov chain. In particular, the risk-free interest rate, the appreciation rate, the volatility and the martingale describing the jump component of the reference investment fund are related to the modulating Markov chain. Two approaches, namely, the regime-switching Esscher transform and the minimal martingale measure, are used to select pricing kernels for the fair valuation. Analytical pricing formulas for the embedded options underlying these products are derived using the inverse Fourier transform. The fast Fourier transform approach is then used to numerically evaluate the embedded options. Numerical examples are provided to illustrate our approach.

1. Introduction

Equity-linked annuities (ELAs) are one of the major innovations in the insurance industry. They provide policyholders with insurance protection as well as investment returns from equity markets. These contracts allow the flexibility to provide both life insurance benefits and guaranteed minimum accumulation benefits. Typically, in an ELA contract, an insurer will make periodic payments to the beneficiary, while the policyholder pays a lump-sum premium at the initiation of the contract. From a policyholder’s perspective, ELAs provide minimum guarantees on downside risk and upside potential profits. The policyholder is also provided with the flexibility to select the composition of an investment portfolio. Furthermore, the tax-deferred feature is another advantage of these products. From the perspective of an insurer, higher insurance fees is a main advantage. In practice, the operational procedure to sell ELAs is comparatively easier. These may explain why both policyholders and insurers prefer these products to other long-term investments with lower yields, including bank accounts, bonds, and so on. Two popular types of ELAs are equity-indexed annuities (EIAs) and variable annuities (VAs) with various embedded guarantees.

The valuation of ELAs, including EIAs and VAs has attracted a considerable interest from both academic researchers and market practitioners. The literature mostly investigate the valuation of ELAs based on the interplay between an option and an ELA (see Boyle and Schwartz, 1977 and Brennan and Schwartz, 1976, 1979). The guaranteed minimum benefit can be viewed as a kind of embedded options. Much attention has been given to the EIAs valuation under the Black–Scholes framework, including Tiong (2000), Lee (2003), etc. Lin and Tan (2003) and Kijima and Wong (2007) investigated the valuation of EIAs with stochastic interest rates and mortality risk, while Qian et al. (2010) considered the EIAs valuation with stochastic mortality rate. Milevsky and Posner (2001) investigated the valuation of guaranteed minimum death benefit (GMDB) in VAs by the risk-neutral pricing theory. Examples of considering the valuation of guaranteed minimum withdrawal benefit (GMWB) in VAs include Milevsky and Salisbury (2006) and Dai et al. (2008). Hardy (2003) presented an overview of various investment guarantees. Bauer et al. (2008) considered a general pricing framework for all types of guarantees in VAs. Siu et al. (2007) and Ng et al. (2011) discussed the valuation of investment guarantees under GARCH-type models.

Regime-switching models are popular and practically useful models in econometrics and finance. This class of models was...
two approaches to selecting pricing kernels. Firstly, we use the generalized version of the regime-switching Esscher transform introduced in Shen et al. (2014) to select an equivalent martingale measure (i.e. risk-neutral probability measure). Then we discuss the selection of an equivalent martingale measure using the minimal martingale measure method. Both approaches allow us to determine a unique equivalent martingale measure and incorporate not only the diffusion risk described by the Brownian motion but the regime-switching risk (or the jump risk) modeled by the Markov chain in the valuation. Under the selected risk-neutral probability measure, we use the inverse Fourier transform to derive integral pricing formulas for the embedded options. The fast Fourier transform (FFT) method is adopted to discretize the integral pricing formulas. Since the double regime-switching model is an extension of the single one, the valuation problem under the single model considered in Fan (2013) may be considered as a particular case of the valuation problem in our current paper. Using the FFT method, we provide the numerical examples to illustrate the valuation of the point-to-point EIAs and GMDB in VAs under both the double regime-switching model and the single regime-switching model as well as document the pricing implications of these two models.

Lin et al. (2009), one of our main references, considered an interesting problem to price annuity guarantees under a regime-switching model. Our paper extends the results of Lin et al. (2009) in the following aspects. Firstly, we consider the valuation of EIAs and VAs under a double regime-switching model in Shen et al. (2014). In addition to the assumption that model parameters are governed by the modulating Markov chain adopted in Lin et al. (2009), we also assume that a jump in the price level of the reference investment fund may occur when the modulating Markov chain switches from one state to another. In other words, the impacts of the regime-switching risk were not considered in Lin et al. (2009). However, the regime-switching risk brought by the state transitions of the underlying economy is difficult, if not impossible, to be diversified. This may suggest that the regime-switching risk may not be ignored. Secondly, we provide two different ways to endogenously determine the regime-switching risk using the generalized regime-switching Esscher transform and the minimal martingale measure approach. In the discussion part of Lin et al. (2009), an exogenous way to quantify the regime-switching risk was provided. However, there may exist more than one solutions for the regime-switching Esscher transform parameters from the given density process described in the discussion of Lin et al. (2009). Other techniques are needed to choose an equivalent martingale measure. In the model we considered here, without imposing other criteria or constraints, a unique pricing kernel can be selected using either the generalized version of regime-switching Esscher transformation or the minimal martingale measure approach. Furthermore, this pricing kernel also provides a quantification for the regime-switching risk. Thirdly, our results may be easier to be extended to a multi-regime case. The analytical pricing formulas, obtained via the FFT approach, look quite neat and the convergence rate of the FFT approach is reasonably fast.

The rest of the paper is organized as follows. The next section presents the model dynamics. In Section 3, we select equivalent martingale measures using the generalized version of the regime-switching Esscher transformation and the minimal martingale measure approach. Section 4 presents the valuation of the point-to-point EIAs and the annual ratchet EIAs. The FFT approach and the Monte Carlo method are applied to calculate the prices of the point-to-point EIAs and the annual ratchet EIAs, respectively. The valuation of the variable annuities with GMDB is considered in Section 5. In Section 6, we give numerical examples to illustrate the valuation of the point-to-point EIAs, the annual ratchet EIAs and VAs with GMDB. Section 7 concludes the paper.
2. The model dynamics

The modeling framework presented in this section resembles that considered in Shen et al. (2014). Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which describes uncertainties attributed to a standard Brownian motion and a Markov chain. We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} := \{\mathcal{F}(t)\} t \in \mathcal{T}$ satisfying the usual conditions of right-continuity and $\mathcal{P}$-completeness. Suppose that $\mathcal{P}$ is a real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model, where $T \ll \infty$. We describe the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $X := \{X(t) \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite-state space $\mathcal{S} := \{s_1, s_2, \ldots, s_N\}$. The states of the chain $X$ are interpreted as different states of an economy or different stages of a business cycle. Without loss of generality, we adopt the canonical state space representation of the chain in Elliott et al. (1994) and identify the states of the chain with a finite set of standard unit vectors $\mathcal{S} := \{e_1, e_2, \ldots, e_N\} \subset \mathbb{R}^N$, where the $j$th component of $e_j$ is the Kronecker delta $\delta_{jl}$, for each $j, l = 1, 2, \ldots, N$.

Define $A := \{a_{jl}\}_{j=1, l=1}^{N} \subset \mathbb{R}^N$ as the rate matrix of the chain $X$ under $\mathcal{P}$, where $a_{jl}$ represents the transition intensity of the chain $X$ from state $e_j$ to state $e_l$. Note that $a_{jl} \geq 0$, for $j \neq l$ and $\sum_{l=1}^{N} a_{jl} = 0$ for each $j, l = 1, 2, \ldots, N$. Then, the following semi-martingale representation is obtained by Elliott et al. (1994):

$$X(t) = X(0) + \int_{0}^{t} AX(s)ds + M(t), \quad t \in \mathcal{T},$$

where $\{M(t) \in \mathcal{T}\}$ is an $\mathcal{R}^N$-valued square-integrable martingale with respect to the filtration generated by the Markov chain $X$.

Suppose $p_{jl}(t)$ is the number of jumps of the chain $X$ from state $e_j$ to state $e_l$ up to time $t$ and $\Phi(t)$ counts the number of jumps of the chain $X$ into state $e_j$ from other states up to time $t$ for each $t \in \mathcal{T}$ and $j, l = 1, 2, \ldots, N$. Denote

$$\Phi(t) := \sum_{j=1, j \neq l}^{N} p_{jl}(t)$$

$$= \sum_{j=1, j \neq l}^{N} \sum_{0 \leq s \leq t} 1 \{X(s) = e_j\} \{X(s), e_l\}$$

$$= \sum_{j=1, j \neq l}^{N} \int_{0}^{t} \{X(s) = e_j\} (AX(s), e_l) ds$$

$$+ \int_{0}^{t} \{X(s) = e_j\} \{dM(s), e_l\} = \Phi(t) + \tilde{\Phi}(t),$$

where

$$\Phi(t) := \sum_{j=1, j \neq l}^{N} a_{jl} \int_{0}^{t} \{X(s) = e_j\} ds,$$

and

$$\tilde{\Phi}(t) := \sum_{j=1, j \neq l}^{N} \int_{0}^{t} \{X(s) = e_j\} \{dM(s), e_l\}.$$

It is easy to see that $\{\Phi(t) | t \in \mathcal{T}\}$ is a martingale. The differential form of the martingale $\{\tilde{\Phi}(t) | t \in \mathcal{T}\}$ can be represented as

$$d\tilde{\Phi}(t) = d\Phi(t) - a_{jl}(t)dt,$$

where $a_{jl}(t) = \sum_{j=1, j \neq l}^{N} a_{jl} \{X(t), e_l\}$, for $l = 1, 2, \ldots, N$. This is a version of the Doob–Meyer decomposition.

We assume that there are two primitive assets, namely, a zero-coupon bond $B$ and an investment fund $S$, in the financial market. The instantaneous market interest rate is assumed to be modulated by the chain $X$ as follows:

$$r(t) := (r, X(t)) = \{r, X(t)\}, \quad t \in \mathcal{T},$$

where $r := (r_1, r_2, \ldots, r_N)' \in \mathbb{R}^N$, with $r_j > 0$ for each $j = 1, 2, \ldots, N$. Here $\mathcal{Y}$ is the transpose of a vector or a matrix $\mathcal{Y}$ and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$. Then the dynamics of the zero-coupon bond $B := \{B(t) \in \mathcal{T}\}$ is given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1.$$

Further, suppose that the appreciation rate and the volatility of the investment fund at time $t$ are also modulated by the Markov chain $X$, i.e.,

$$\mu(t) := \{\mu, X(t)\}, \quad \sigma(t) := \{\sigma, X(t)\}, \quad t \in \mathcal{T},$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{R}^N$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N$, with $\mu_j > 0$ and $\sigma_j > 0$ for each $j = 1, 2, \ldots, N$.

Let $\mathcal{W} := \{W(t) \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. To simplify our discussion, we suppose that $W$ and $X$ are stochastically independent under $\mathcal{P}$. Under the real-world probability measure $\mathcal{P}$, the price process of the investment fund is governed by the following double regime-switching model:

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)d\mathcal{W}(t)$$

$$+ \sum_{j=1}^{N} (e^{\beta_j(t)} - 1) d\tilde{\Phi}(t), \quad S(0) = S_0 > 0,$$

where $\beta_j(t) := (\beta_1, X(t))$ and $\beta_j := (\beta_1, \beta_2, \ldots, \beta_N)' \in \mathbb{R}^N$.

Write $\beta := (\beta_1, \beta_2, \ldots, \beta_N)' \in \mathbb{R}^{N \times N}$. Here the jump ratio in the investment fund price level when the chain transits from state $e_j$ to state $e_l$ is given by

$$\left\{ \begin{array}{ll}
\exp(\beta_{jl}) - 1, & j \neq l, \\
0, & j = l.
\end{array} \right.$$

Define $Y(t) := \log(S(t)/S_0)$ as the logarithmic return of the investment fund during the time horizon $[0, t]$, for each $t \in \mathcal{T}$. Then by Itô’s differentiation rule, it is easy to see that

$$dY(t) = \left[ \mu(t) - \frac{1}{2} \sigma^2(t) - \sum_{j=1}^{N} (e^{\beta(t)} - 1 - \beta_j(t) - a_{jl}(t)) \right] dt$$

$$+ \sigma(t)d\mathcal{W}(t) + \sum_{j=1}^{N} \beta_j(t) d\tilde{\Phi}(t), \quad t \in \mathcal{T}.$$
3. Equivalent martingale measures

It is well known that the financial market with regime-switching is incomplete and there exist more than one equivalent martingale measures in the incomplete market. Various approaches have been proposed in the literature for the selection of an equivalent martingale measure in an incomplete market. In this section, we apply the generalized regime-switching Esscher transform and the minimal martingale measure method to determine an equivalent martingale measure. We present the (local) martingale conditions in the two cases. Gerber and Shiu (1994) pioneered the Esscher transformation approach to value options in incomplete markets. This approach provides market practitioners with a convenient way to determine a pricing kernel and can be justified by the maximization of an expected power utility of an economic agent. Bühmann et al. (1996) and Kallsen and Shiryaev (2002) extended the Esscher transform to a generalized one for a general semimartingale. Elliott and Siu (2013) and Siu (2014) adopted the generalized Esscher transform in Bühmann et al. (1996) and Kallsen and Shiryaev (2002) to determine a pricing kernel for a hidden Markov-modulated pure-jump asset price process and a hidden Markov-modulated jump-diffusion model, respectively. Elliott et al. (2005) introduced a regime-switching Esscher transform to price options under regime-switching models, which was further justified by Siu (2008, 2011). Shen et al. (2014) first considered a generalization of the regime-switching Esscher transform to select an equivalent martingale measure under the double regime-switching model. The discussions of the regime-switching Esscher transform to be presented below follow those in Kallsen and Shiryaev (2002), Elliott and Siu (2013) and Shen et al. (2014). In addition, (locally) risk-minimization method was applied to identify an equivalent martingale measure, referred to as the minimal martingale measure, by Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996). Under a regime-switching jump-diffusion model, Su et al. (2012) considered the selection of minimal martingale measure and its use for valuing European options.

3.1. Esscher transformed equivalent martingale measure

In this subsection, we discuss how to select a pricing kernel using a generalized version of regime-switching Esscher transform. Let \( \mathcal{L}(Y) \) be the space of all processes \( \theta := \{ \theta(t) \mid t \in \mathcal{T} \} \) such that

1. For each \( t \in \mathcal{T} \), \( \theta(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_N(t)) \) in \( \mathbb{R}^N \);
2. \( \theta \) is integrable with respect to \( Y \) in the sense of stochastic integration.

For each \( \theta \in \mathcal{L}(Y) \), we denote the stochastic integral of \( \theta \) with respect to \( Y \) as

\[
(\theta \cdot Y)(t) := \int_0^t \theta(s) dY(s), \quad t \in \mathcal{T},
\]

where \( \theta \) is called the Esscher transform parameter.

For each \( \theta \in \mathcal{L}(Y) \), define a \( \mathcal{G} \)-adapted exponential process \( D^\theta := \{ D^\theta(t) \mid t \in \mathcal{T} \} \) as follows:

\[
D^\theta(t) := \exp((\theta \cdot Y)(t)),
\]

where

\[
H^\theta(t) := \int_0^t \theta(s) \left[ \frac{\mu(s)}{\sigma(s)} - \frac{1}{2} \sigma^2(s) \right] ds + \int_0^t \frac{1}{2} \theta^2(s) \sigma^2(s) ds - \sum_{i=1}^N \int_0^t \left[ e^{\theta(s)} - 1 - \theta(s) \beta_i(s) \right] a_i(s) ds + \int_0^t \frac{1}{2} \theta^2(s) \sigma^2(s) ds + \int_0^t \left[ 1 - \theta(s) \beta_i(s) \right] a_i(s) ds.
\]

is a \( \mathcal{G} \)-adapted process. Then \( D^\theta \) is the Doléans-Dade stochastic exponential of \( H^\theta \), i.e.,

\[
D^\theta(t) = \mathcal{E}(H^\theta(t)), \quad t \in \mathcal{T}.
\]

The Laplace cumulant process \(^1\) of the stochastic integral process \( (\theta \cdot Y) \) under \( \mathcal{P} \) is given by

\[
\mathcal{E}(\mathcal{M}^\theta(t)) = 1 + \int_0^t \mathcal{E}(\mathcal{M}^\theta(s)) d\mathcal{M}^\theta(s) = \exp(\mathcal{M}^\theta(t)), \quad t \in \mathcal{T}.
\]

The second equality is due to the fact that \( \{ \mathcal{M}^\theta(t) \mid t \in \mathcal{T} \} \) is a finite variation process. Then, the logarithmic transform \( \mathcal{M}^\theta := \{ \mathcal{M}^\theta(t) \mid t \in \mathcal{T} \} \) of \( \mathcal{M}^\theta(t) \), for each \( \theta \in \mathcal{L}(Y) \), is given by

\[
\mathcal{M}^\theta(t) := \log(\mathcal{E}(\mathcal{M}^\theta(t))) = \mathcal{M}^\theta(t), \quad t \in \mathcal{T}.
\]

Let \( \Lambda^\theta := \{ \Lambda^\theta(t) \mid t \in \mathcal{T} \} \) be a \( \mathcal{G} \)-adapted process associated with \( \theta \in \mathcal{L}(Y) \) defined by:

\[
\Lambda^\theta(t) := \exp((\theta \cdot Y)(t) - \mathcal{M}^\theta(t)), \quad t \in \mathcal{T}.
\]

Then from Eqs. (3) and (4), we obtain

\[
\Lambda^\theta(t) := \exp\left[ \int_0^t \theta(s) \sigma(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) \sigma^2(s) ds + \int_0^t \sum_{i=1}^N \theta(s) \beta_i(s) d\tilde{\Phi}_i(s) \right].
\]

Since \( \mathcal{L}(Y) \) is an exponential process compensated by its modified Laplace cumulant process, it is a \( (\mathcal{G}, \mathcal{P}) \)-(local)-martingale. Indeed, this can also be checked easily via Itô’s differentiation rule. It is assumed that \( \theta \in \mathcal{L}(Y) \) satisfying certain standard technical conditions so that \( \Lambda^\theta \) is a \( (\mathcal{G}, \mathcal{P}) \)-martingale.

For each \( \theta \in \mathcal{L}(Y) \), we define a new probability measure \( \mathcal{Q}^\theta \) equivalent to \( \mathcal{P} \) on \( \mathcal{G}(T) \) by a generalized version of the regime-switching Esscher transform \( \Lambda^\theta(T) \) as follows:

\[
\frac{d\mathcal{Q}^\theta}{d\mathcal{P}} \bigg|_{\mathcal{G}(T)} := \Lambda^\theta(T).
\]

The following lemmas, namely Lemmas 3.1–3.3, follow from Lemmas 2–4 in Shen et al. (2014). So we state the results here without giving the proof.

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1 For more discussion on the Laplace cumulant process as well as the Esscher transform given below, interested readers can refer to Kallsen and Shiryaev (2002) and Elliott and Siu (2013).
Lemma 3.1. Define the discounted price of the investment fund as follows:
\[
\tilde{S}(t) := \exp \left\{ - \int_0^t r(s) \, ds \right\} S(t), \quad \tilde{S}(0) = S_0, \quad t \in \mathcal{T}.
\]
According to Eq. (1) and Itô’s differentiation rule, we can get the dynamic of the discounted price process as follows
\[
\tilde{S}(t) = S_0 + \int_0^t \tilde{S}(s)(\mu(s) - r(s)) \, ds + \int_0^t \tilde{S}(s) \sigma(s) \, dW(s)
+ \int_0^t \tilde{S}(s) \sum_{l=1}^N (e^{\phi_l(t-)} - 1) \, d\tilde{\Phi}_l(s).
\]
Then the discounted price process \(\tilde{S}(t)\) is a \((\mathcal{G}, \mathbb{Q}^\theta)\)-martingale if and only if the Esscher transform parameter \(\theta\) satisfies the following equation:
\[
\mu(t) - r(t) + \theta(t) \sigma^2(t) + \sum_{l=1}^N (e^{\phi_l(t-)} - 1)(e^{\beta_l(t)} - 1) \alpha_l(t) = 0.
\] (6)

Remark 3.1. When \(X(t) = e_j\), for each \(j = 1, 2, \ldots, N\), Eq. (6) becomes
\[
\mu_j - r_j + \theta \sigma_j^2 + \sum_{l=1, l \neq j}^N (e^{\phi_l(t)} - 1)(e^{\beta_l(t)} - 1) \alpha_{jl} = 0.
\] (7)
As discussed in Shen and Siu (2013b), the regime-switching Esscher transform parameter is uniquely determined when the model parameters are given. This means that a unique equivalent martingale measure can be determined via the Esscher transformation approach. This is why we say that the market price of regime-switching risk is endogenously determined by the regime-switching Esscher transform.²

Lemma 3.2. For each \(t \in \mathcal{T}\), let
\[
W^\theta(t) := W(t) - \int_0^t \theta(s) \sigma(s) \, ds,
\]
and
\[
\dot{\Phi}_l(t) := \Phi_l(t) - \dot{\phi}_l(t) = \Phi_l(t) - \int_0^t a_l(s-), ds,
\]
where
\[
\phi_l(t) := \int_0^t e^{\phi_l(s)} \, ds,
\]
and
\[
a_l(t) := \int_0^t e^{\phi_l(s)} \sigma(s) \, dW(s).
\]
Then \(W^\theta(t) := \{W^\theta(t) | t \in \mathcal{T}\}\) is a standard Brownian motion under \(\mathcal{Q}^\theta\), and \(\dot{\Phi}_l(t) := \{\dot{\phi}_l(t) | t \in \mathcal{T}\}\) is an \((\mathbb{F}^\mathcal{T}, \mathcal{Q}^\theta)\)-martingale, for each \(l = 1, 2, \ldots, N\).

Furthermore, suppose \(\mathcal{N}^\theta\) is an \((N \times N)\)-matrix with the following entries:
\[
a_{jj}^\theta := \begin{cases} e^{\phi_j(t)} \alpha_j, & j \neq l, \\ - \sum_{l=1, l \neq j}^N e^{\phi_l(t)} \alpha_{jl}, & j = l. \end{cases}
\]
Then the chain \(X\) has the following semimartingale decomposition under \(\mathcal{Q}^\theta\)
\[
X(t) = X(0) + \int_0^t A_t^\theta X(s) \, ds + M^\theta(t),
\]
where \(M^\theta := \{M^\theta(t) | t \in \mathcal{T}\}\) is an \(\mathbb{R}^N\)-valued, \((\mathbb{F}^\mathcal{T}, \mathcal{Q}^\theta)\)-martingale.

Lemma 3.3. Under \(\mathcal{Q}^\theta\), the return process is given by:
\[
dY(t) = \left[ (r(t) - \frac{1}{2} \sigma^2(t) - \sum_{l=1}^N e^{\phi_l(t)} \beta_l(t) \right] dt
+ \sigma(t-) \, dW^\theta(t)
+ \sum_{l=1}^N \beta_l(t-) \, d\dot{\Phi}_l(t), \quad t \in \mathcal{T}.
\] (8)

3.2. Minimal martingale measure

In this subsection, we consider how to determine the minimal martingale measure following Föllmer and Schweizer (1991). Being a semimartingale, the discounted price process \(\tilde{S}(t)\) can be written as
\[
\tilde{S}(t) = S_0 + \Gamma(t) + \Upsilon(t),
\]
where
\[
\Gamma(t) = \int_0^t \tilde{S}(s) \, \theta(s) \, r(s) \, ds
\]
and
\[
\Upsilon(t) = \int_0^t \tilde{S}(s) \, \sigma(s) \, dW(s) + \int_0^t \tilde{S}(s) \sum_{l=1}^N (e^{\phi_l(s)} - 1) \, d\tilde{\Phi}_l(s).
\]
The following definition of the minimal martingale measure was given in Föllmer and Schweizer (1991).

Definition 3.1. An equivalent martingale measure \(\mathcal{Q}^m\) with respect to \(\mathcal{P}\) is called the minimal martingale measure if the following conditions hold:
1. \(\mathcal{Q}^m = \mathcal{P}\) on \(\mathcal{F}_0\);
2. Define \(L\) as a square-integrable \(\mathcal{P}\)-martingale, orthogonal to \(\Upsilon\) under the original measure. After the measure change, \(L\) remains a martingale under \(\mathcal{Q}^m\).

Theorem 3.1 presents the expression of the minimal martingale measure.

Theorem 3.1. Define a new probability measure \(\mathcal{Q}^m\) equivalent to \(\mathcal{P}\) on \(\mathcal{F}_1(\Gamma)\) by the following Radon–Nikodym derivative:
\[
\frac{d\mathcal{Q}^m}{d\mathcal{P}} |_{\mathcal{F}_1(\Gamma)} := Z(\Gamma),
\]
where
\[
Z(t) = \exp \left\{ \int_0^t \frac{-(\mu(s) - r(s))\sigma(s)}{\sigma^2(s) + \sum_{i=1}^N (e^{\theta_i(s)} - 1)^2 a_i(s)} \, dW(s) \right. \\
- \frac{1}{2} \int_0^t \frac{(\mu(s) - r(s))^2\sigma^2(s)}{(\sigma^2(s) + \sum_{i=1}^N (e^{\theta_i(s)} - 1)^2 a_i(s))} \, ds \\
+ \int_0^t \sum_{i=1}^N \frac{(\mu(s) - r(s))(e^{\theta_i(s)} - 1)}{\sigma^2(s) + \sum_{i=1}^N (e^{\theta_i(s)} - 1)^2 a_i(s)} \, a_i(s) \, ds \\
+ \int_0^t \sum_{i=1}^N \ln \left( 1 - \frac{(\mu(s) - r(s))(e^{\theta_i(s)} - 1)}{\sigma^2(s) + \sum_{i=1}^N (e^{\theta_i(s)} - 1)^2 a_i(s)} \right) \, d\Phi_i \right\}. \tag{9}
\]

Furthermore, under the following assumptions,
\[
\frac{(\mu(t) - r(t))(e^{\theta(t)} - 1)}{\sigma^2(t) + \sum_{i=1}^N (e^{\theta_i(t)} - 1)^2 a_i(t)} < 1, \quad \text{a.s. for } t \in \mathcal{T},
\]
then \( \tilde{Q}^m \) is the unique minimal martingale measure.

**Proof.** The proof of Theorem 3.1 resembles to Theorem 1 in Föllmer and Schweizer (1991) and Theorem 3.1 in Su et al. (2012). Here we only present the key steps of the proof.

- **Uniqueness.**
  Suppose \( \tilde{Q} \) is a minimal martingale measure. Define \( \tilde{Z}(t) \) as
  \[
  \tilde{Z}(t) = E \left[ \frac{d\tilde{Q}}{d\tilde{P}} \right] g(t).
  \]
  Then, according to the Kunita–Watanabe decomposition,
  \[
  \tilde{Z}(t) = \tilde{Z}(0) + \int_0^t \tilde{\theta}(s) d\mathcal{Y}(s) + L(t), \quad \tilde{Z}(0) = 1.
  \]
  Here, \( L \) is a square-integrable martingale and orthogonal to \( \mathcal{Y} \) under \( \tilde{P} \) and \( \tilde{\theta} \) is a predictable process satisfying \( E \left[ \int_0^T \tilde{\theta}^2(s) d\langle \mathcal{Y}, \mathcal{Y} \rangle(s) \right] < \infty \). Using Girsanov’s theorem (see, for example, Proter, 2004), it is easy to see that the predictable process of finite variation under the new probability measure \( \tilde{Q} \) can be written as
  \[
  \tilde{\Gamma}(t) = -\int_0^t \frac{1}{Z(s-)} \tilde{d}(\mathcal{Y}, \tilde{Z})(s),
  \]
  where \( \tilde{Z}(t) \) is positive for each \( t \in \mathcal{T} \) by definition. According to the definition of the minimal martingale measure, \( (L, \tilde{Z} \equiv \{L, \tilde{Z}\}) = 0 \). Hence \( \tilde{Z}(t) = 1 + \int_0^t \tilde{\theta}(s) d\mathcal{Y}(s) + \tilde{d}\Gamma(t) = -\frac{\partial \tilde{Z}(t)}{\partial s} \tilde{d}(\mathcal{Y}, \tilde{Z})(t) \). Then
  \[
  \tilde{Z}(t) = 1 - \int_0^t \tilde{Z}(s-) \frac{d\Gamma(s)}{d(\mathcal{Y}, \mathcal{Y})(s)} \, d\mathcal{Y}(s). \tag{10}
  \]

Since there exists a unique solution of Eq. (10), we obtain \( \tilde{Z} = Z \) and the uniqueness of the minimal martingale measure if there exists a minimal martingale measure.

- **Existence.**
  By the Doob–Meyer decomposition of \( \tilde{S} \) and the Doléans-Dade exponential, we can solve the explicit form of \( Z(t) \), which is given by Eq. (9). Note that \( Z \) is a square-integrable martingale under \( \tilde{P} \) when Assumptions 1 and 2 hold. Consequently, if \( \tilde{Q}^m \) exists, \( \{Z(t) \mid t \in \mathcal{T}\} \) is a square-integrable martingale under the original measure. Conversely, if \( Z(t) \) is given by Eq. (9), we intend to verify the martingale measure \( \tilde{Q}^m \) is the minimal martingale measure. Define \( L \) as a square-integrable \( \mathcal{P} \)-martingale orthogonal to \( \mathcal{Y} \). Since \( Z \) is the solution of Eq. (10), \( (L, Z) = 0 \), verifying that \( L \) is a local martingale under \( \tilde{Q}^m \). Note that \( L \) is a square-integrable martingale. So \( L \) is a martingale under \( \tilde{Q}^m \) and \( \tilde{Q}^m \) is the minimal martingale measure. \( \Box \)

Similar to discussions in Section 3.1, we also present the (local)-martingale condition and the dynamics of the Brownian motion, the Markov chain and the logarithmic return process under the minimal martingale measure. The derivations of Lemmas 3.4–3.6 resemble to those of Lemmas 3.1–3.3. So we state the results without giving the proofs.

**Lemma 3.4.** The (local)-martingale condition is given by
\[
\mu(t) - r(t) - \frac{(\mu(t) - r(t))\sigma^2(t)}{\sigma^2(t) + \sum_{i=1}^N (e^{\theta_i(t)} - 1)^2 a_i(t)} \geq 0. \tag{11}
\]

**Remark 3.2.** It is not difficult to see the martingale condition (11) always holds no matter what values \( \mu(t), r(t), \sigma(t) \) and \( \beta(t) \) take, for each \( l = 1, 2, \ldots, N \) and \( t \in \mathcal{T} \).

**Lemma 3.5.** For each \( t \in \mathcal{T} \), let
\[
W^m(t) = W(t) + \int_0^t \frac{(\mu(s) - r(s))\sigma(s)}{\sigma^2(s) + \sum_{i=1}^N (e^{\theta_i(s)} - 1)^2 a_i(s)} \, ds,
\]
and
\[
\tilde{\Phi}_i^m(t) := \Phi_i(t) - \tilde{\Phi}_i^m(t) = \Phi_i(t) - \int_0^t a_i(s-) \, ds,
\]
where
\[
a_i(t) := 1 - \frac{(\mu(t) - r(t))(e^{\theta_i(t)} - 1)}{\sigma^2(t) + \sum_{i=1}^N (e^{\theta_i(t)} - 1)^2 a_i(t)} a_i(t).
\]

Then \( W^m := \{W^m(t) \mid t \in \mathcal{T}\} \) is a standard Brownian motion under \( \tilde{Q}^m \), and \( \tilde{\Phi}_i^m := \{\tilde{\Phi}_i^m(t) \mid t \in \mathcal{T}\} \) is an \( (\mathbb{R}^N, \tilde{Q}^m) \)-martingale, for each \( l = 1, 2, \ldots, N \).

**Lemma 3.6.** Under \( \tilde{Q}^m \), the return process is given by:
\[
dY(t) = \left[ r(t) - \frac{1}{2} \sigma^2(t) \right] dt - \sum_{i=1}^N (e^{\theta_i(t)} - 1 - \beta_i(t)) a_i(t-) dt \\
+ \sigma(t-)dW(t) + \sum_{i=1}^N \beta_i(t-)d\tilde{\Phi}_i^m(t), \quad t \in \mathcal{T}. \tag{12}
\]

**Remark 3.3.** If we assume the jump ratio in the investment fund price level when the chain transits from state \( e_i \) to state \( e_j \) becomes
zero, the minimal martingale measure \( \mathcal{Q}^m \) is given by
\[
\frac{d\mathcal{Q}^m}{d\mathcal{P}} \bigg|_{\mathcal{F}_t} = \exp \left\{ \int_0^t \left( \frac{\mu(s) - r(s)}{\sigma^2(s)} - \frac{1}{2} \int_0^s \frac{1}{\sigma^2(s)} \left( \frac{\mu(s) - r(s)}{\sigma^2(s)} \right)^2 ds \right) dW(s) \right\}.
\]

On the other hand, with this assumption, the Radon–Nikodym derivative of the regime-switching Esscher transform becomes
\[
\frac{d\mathcal{Q}^\theta}{d\mathcal{P}} \bigg|_{\mathcal{F}_t} = \exp \left\{ \int_0^t \theta(s)\sigma(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s)\sigma^2(s) ds \right\}.
\]

According to the martingale condition, the regime-switching Esscher transform parameter can be determined uniquely by
\[
\hat{\theta}(t) = \frac{r(t) - \mu(t)}{\sigma^2(t)}.
\]

Consequently, in this case, the Esscher transformed equivalent martingale measure is consistent with the minimal martingale measure. This is the same as the regime-switching Esscher transform in Elliott et al. (2005). This result is also illustrated in our numerical examples.

**Remark 3.1:** Compared with Yuen and Yang (2009), the main advantage of both the regime-switching Esscher transform and the minimal martingale measure method is that these two approaches can determine a unique pricing kernel, while Yuen and Yang (2009) cannot select a unique pricing kernel. The derivation of the local-risk-minimizing measure under a Markov regime-switching jump-diffusion model, where the jump component is relating to the regime-switching Esscher transform in Elliott et al. (2011).

### 4. Valuation of equity-indexed annuities

In this section, we will discuss the valuation of equity-indexed annuities with guarantees, including the point-to-point EIAs and the annual ratchet EIAs, under a double regime-switching model. For the point-to-point EIAs, analytical pricing formulas are derived by the inverse Fourier transform and the FFT method is applied to calculate the prices of the point-to-point EIAs. For the annual ratchet EIAs, the valuation relies on numerical methods. Although the derivations here involve some standard mathematical techniques, the analytical pricing formulas derived might hopefully be of some practical values. The proofs presented in this section are standard and do not involve new mathematical results. They are presented here for the ease of exposition.

For notational simplicity, we denote \( \mathcal{Q}^* \) as the risk-neutral martingale measure selected by either the regime-switching Esscher transform or the minimal martingale measure in Section 3. \( \mathbb{E}^*[\cdot] \) is the expectation under the risk-neutral probability measure \( \mathcal{Q}^* \). Under \( \mathcal{Q}^* \), the dynamics of the return process and the Markov chain are, respectively, given by:
\[
dY(t) = \left[ r(t) - \frac{1}{2} \sigma^2(t) \right] dt - \sum_{i=1}^{N} \left( e^{\beta_i(t)-1} - \beta_i(t) \right) d\tilde{\Phi}_i(t), \quad t \in \mathcal{T},
\]
and
\[
X(t) = X(0) + \int_0^t \mathbf{A}'X(s) ds + \mathbf{M}'(t), \quad t \in \mathcal{T},
\]
where \( \mathbf{W}^* := \{ W^*(t) \mid t \in \mathcal{T} \} \) is a standard Brownian motion under \( \mathcal{Q}^* \); \( \Phi^*_i := \{ \Phi^*_i(t) \mid t \in \mathcal{T} \} \) is an \( \mathbb{R}^k \)-valued martingale; \( \mathbf{A}^\star := \{ a_{ij} \}_{i,j=1,2,...,N} \) is the rate matrix of the Markov chain under \( \mathcal{Q}^* \); \( \mathbf{M}^\star := \{ M^\star(t) \mid t \in \mathcal{T} \} \) is an \( \mathbb{R}^k \)-valued, \( \mathbb{R}^k \)-valued martingale.

#### 4.1. Valuation of point-to-point EIAs

In this subsection we discuss the valuation of the point-to-point EIAs, the simplest type of EIAs. In year \( t \), the payoff of the point-to-point EIA \( G_{pp}(t) \) is given by
\[
G_{pp}(t) := \max \{ \min(\omega(Y(t)), v), e^{\alpha t}, e^{\beta t} \},
\]
where \( \omega \) is the participation rate, \( z \) is the continuously compounded minimum guarantee rate and \( \gamma \) is the continuously compounded cap rate.

If the policyholder survives until the maturity of the contract, the policyholder will receive \( G_{pp}(T) \) at time \( T \). If the policyholder dies during the time period \( (t, t+1] \), for \( t = 1, 2, \ldots, T \), the beneficiary will receive \( G_{pp}(t) \) at \( t+1 \). Consequently, the payoff of a point-to-point EIA with mortality risk is given by
\[
\begin{align*}
G_{pp}(t), & \quad t \leq T(x) < t+1, \\
G_{pp}(T), & \quad T(x) > T(t),
\end{align*}
\]
where \( T(x) \) is the future lifetime of a policyholder with purchased age \( x \). Suppose \( T(x) \) is independent of the price processes in the financial market. \( r_x \) denotes the probability that the aged-\( x \) policyholder will survive until the contract expires at \( T \) and \( q_x \) represents the probability that the aged-\( x \) policyholder dies in \( t \) years. The mathematical definitions are given by \( r_x = P(T(x) \geq T) \) and \( q_x = 1 - r_x = P(T(x) \leq T) \), respectively.

Define the conditional characteristic function of \( Y(t) \) given \( \mathcal{F}_t \) under \( \mathcal{Q}^* \) as:
\[
\psi_{Y(t)}(0, t, v) := \mathbb{E}^{\mathcal{Q}^*} \left[ e^{ivY(t)} \mid \mathcal{F}_t \right],
\]
and the unconditional discounted characteristic function of \( Y(t) \) under \( \mathcal{Q}^* \) as:
\[
\widehat{\psi}_{Y(t)}(0, t, v) := \mathbb{E}^*[\exp \left\{ -\int_0^t r(s) ds \right\} \psi_{Y(t)}(0, t, v)].
\]

The following lemma is standard. It resembles to Theorem 1 in Shen et al. (2014), (see also Proposition 3.1 in Fan et al., 2014 and Theorem 4.1 in Shen and Siu, 2013a).

**Lemma 4.1.** Under the risk-neutral probability measure \( \mathcal{Q}^* \), the time-zero value of a European call option with maturity time \( t > 0 \) and payoff \( \max(e^{\alpha Y(t)} - e^{\beta t}) \) is given by
\[
C(0, t, k) := \mathbb{E}^{\mathcal{Q}^*} \left[ \exp \left\{ -\int_0^t r(s) ds \right\} \left( e^{\alpha Y(t)} - e^{\beta t} \right) \right] = \frac{e^{\alpha k} \psi(0, t, v)}{\pi} \int_0^\infty e^{-i k \psi} (0, t, v) dv,
\]
where
\[
\psi(0, t, v) = \frac{\mathbb{E}^{\mathcal{Q}^*} \left[ \exp \left( \text{diag}(g(v - i(\alpha + 1)\omega + B^*t)) \right) \right] \mathbf{1}}{\alpha^2 + \beta^2 - \alpha^2 - \beta^2}.
\]
with \( g(v) := (g_1(v), g_2(v), \ldots, g_N(v))^\prime \) and \( B^* = [b_{ij}]_{i=1,2,...,N} \) given by
\[
g_{ij}(v) := -r_j + iv(r_j - \frac{1}{2} \sigma_j^2 - \frac{1}{2} \hat{\nu}^2 \sigma_j^2) + \sum_{l=1}^{N} \left( e^{i \beta l} - 1 - iv(e^{i \beta l} - 1) \right) a_{ij}^l,
\]
and
\[
\psi(0, t, v) = \left[ \mathbb{E}^{\mathcal{Q}^*} \left[ \text{diag}(g(v - i(\alpha + 1)\omega + B^*t)) \right] \right] \mathbf{1}.
\]
and
\[ b_j^\ast = \begin{cases} e^{i\bar{\nu}_j}a_j^\ast, & j \neq l, \\ - \sum_{l=1, l \neq j} e^{i\bar{\nu}_j}a_l^\ast, & j = l. \end{cases} \quad (17) \]

**Proof.** The proof is standard. We include it here for the sake of completeness. Write \( R_t := \int_0^t r(s)ds \). Let \( F_{Y(t)\mathcal{F}X(t)}(y) \) be the conditional distribution function of \( Y(t) \) given \( \mathcal{F}X(t) \) under \( \mathcal{Q}^* \). Following Carr and Madan (1999) and Liu et al. (2006), let \( \alpha \) be the damping coefficient, we can calculate the Fourier transform of the dampened call option price as follows:

\[
\psi(0, t, v) := \int_R e^{i k}e^{oK}C(0, t, k)dk \\
= \int_R e^{i k}e^{oK}E\left[ e^{oY(t)} - e^{\bar{\nu}_j^*} \right]dk \\
= E^\ast \left[ \int_R e^{i k}e^{oK}E\left[ e^{oY(t)} - e^{\bar{\nu}_j^*} \right] \right] \\
= E^\ast \left[ \int_R e^{-k}e^{\alpha+i\nu}\int_0^\infty (e^{\alpha+i\nu})^{-\nu}F_{Y(t)\mathcal{F}X(t)}(dy)dk \right] \\
= E^\ast \left[ \int_R e^{-k}e^{\alpha+i\nu}\int_0^\infty (e^{\alpha+i\nu})^{-\nu}F_{Y(t)\mathcal{F}X(t)}(dy) \right] \\
= E^\ast \left[ \int_R e^{-k}e^{\alpha+i\nu} \left( \frac{1}{\alpha+i\nu} - \frac{1}{1+\alpha+i\nu} \right) \times F_{Y(t)\mathcal{F}X(t)}(dy) \right] \\
= \frac{E^\ast\left[ e^{\nu_Y(t)}|_{X(t)} \left( (1-i(\alpha+1))\omega \right) \right]}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)v} \\
= \frac{\tilde{\varphi}_{Y(t)}(0, t, (\nu-i(\alpha+1)))}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)v}. \quad (18) \\
\]

Therefore, we can derive the value of the Fourier transform of the option if we know the formulas of the unconditional, discounted characteristic function of \( Y(t) \) under \( \mathcal{Q}^* \). To make our paper self-contained, we briefly discuss the derivation. Detailed proof can be referred to Lemma 6 in Shen et al. (2014). Applying Itô’s differentiation rule to \( e^{oY(t)} \) gives

\[
d e^{oY(t)} = e^{oY(t)} \left[ i\nu r(t) \left( \frac{1}{2} \sigma^2(s) \right) \right. \\
- \frac{1}{2} \alpha^2 \sigma^2(s) + \sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) a_l^\ast ds + i\nu \sigma(s)dW^\ast(s) \\
+ \sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) d\tilde{\varphi}_l^\ast(s). \]

Then

\[
d\tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v) = \tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v) \left[ -r(s) + i\nu r(s) - \frac{1}{2} \sigma^2(s) - \frac{1}{2} \nu^2 \sigma^2(s) + \sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) a_l^\ast(s) \right] ds \\
+ \sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) d\tilde{\varphi}_l^\ast(s). \]

Define

\[
\mathbf{h}(s, v) := \mathbf{X}(s)\tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v), \quad s \in \mathcal{T}. \\
\]

Using the stochastic integration by parts yields

\[
d\mathbf{h}(s, v) = \left( \text{diag}(\mathbf{g}(v)) + \mathbf{A}^\ast \right)\mathbf{h}(s, v)ds \\
+ \mathbf{h}(s, v) \sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) d\tilde{\varphi}_l^\ast(s) \\
+ \tilde{\varphi}_{Y(t)\mathcal{F}X(t)}(0, s, v)dm^\ast(s) \\
+ \Delta \mathbf{X}(s)\Delta \tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v). \quad (19) \\
\]

Denote

\[
\mathbf{A}_0^\ast := \mathbf{A}^\ast - \text{diag}(a_{11}^\ast, a_{22}^\ast, \ldots, a_{NN}^\ast), \\
\mathbf{B}_0^\ast := \mathbf{B}^\ast - \text{diag}(b_{11}^\ast, b_{22}^\ast, \ldots, b_{NN}^\ast), \\
\mathbf{D}_0 := \{d_{ij}\}_{i=1,2,\ldots,N} - \text{diag}(d_{11}, d_{22}, \ldots, d_{NN})', \\
\text{with } d_{ij} = b_{ij}^\ast/q_{ij}^\ast, \text{ for each } i, j = 1, 2, \ldots, N. \]

Note that

\[
\sum_{l=1}^N \left( e^{i\bar{\nu}_l^*} - 1 \right) d\tilde{\varphi}_l^\ast(s) = (\mathbf{D}_0\mathbf{X}(s) - 1 + \mathbf{X}(s))d\tilde{\varphi}^\ast(s), \\
\]

and

\[
\mathbf{X}(s) = \mathbf{X}(0) + \int_0^s \left( 1 - \mathbf{X}(u-1) \right)d\Phi^\ast(u), \\
\]

where \( \mathbf{1} := (1, 1, \ldots, 1)^\prime \in \mathbb{R}^N, \Phi^\ast := (\Phi_1^\ast, \Phi_2^\ast, \ldots, \Phi_N^\ast)^\prime \subseteq \mathbb{R}^N, \Phi^\prime := (\Phi_1^\prime, \Phi_2^\prime, \ldots, \Phi_N^\prime)^\prime \subseteq \mathbb{R}^N \text{ and } \mathbf{I} \text{ is an } (N \times N) \text{-identity matrix}. \\
\]

Then it is easy to check that

\[
\Delta \mathbf{X}(s)\Delta \tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v) \\
= \left( 1 - \mathbf{X}(s)\mathbf{1} \right) \Delta \Phi^\ast(s)(\mathbf{D}_0\mathbf{X}(s) - 1 + \mathbf{X}(s)) \Delta \Phi^\prime(s) \\
\times \tilde{\varphi}_{Y(t)|\mathcal{F}X(t)}(0, s, v) \\
= \left( 1 - \mathbf{X}(s)\mathbf{1} \right) \text{diag}(d\tilde{\varphi}^\ast(s))(\mathbf{D}_0\mathbf{X}(s) - 1 + \mathbf{X}(s))d\tilde{\varphi}^\ast(s) \\
+ (\mathbf{B}^\ast - \mathbf{A}^\ast)\mathbf{h}(s, v)ds. \\
\]

Then taking expectation on both sides of (19) under \( \mathcal{Q}^* \) gives:

\[
dE^\ast[\mathbf{h}(s, v)] = \left( \text{diag}(\mathbf{g}(v)) + \mathbf{B}^\ast \right)E^\ast[\mathbf{h}(s, v)]ds. \\
\]

Solving gives

\[
E^\ast[\mathbf{h}(t, v)] = \mathbf{X}(0)\exp\left[ \text{diag}(\mathbf{g}(v)) + \mathbf{B}^\ast \right]t. \\
\]

Consequently,

\[
\tilde{\varphi}_{Y(t)}(0, t, v) = \left[ E^\ast[\mathbf{h}(t, v)], \mathbf{1} \right] \\
= \left[ \mathbf{X}(0)\exp\left[ \text{diag}(\mathbf{g}(v)) + \mathbf{B}^\ast \right] \mathbf{t}, \mathbf{1} \right]. \\
\]

Adopting the inverse Fourier transform, the analytical pricing formula for the European call option is obtained. \( \square \)

The proposition presents an analytical valuation formula for the EIA contract.

**Proposition 4.1.** Suppose an aged-x policyholder buys a T-maturity point-to-point EIA contract. Then the time-zero value of the EIA contract \( V_{pp}(0, T, \alpha, \gamma, z) \) is given by

\[
V_{pp}(0, T, \alpha, \gamma, z) = \sum_{t=1}^T \sum_{\alpha=1}^T E^\ast\left[ e^{-\int_0^t r(s)ds}G_{pp}(t)\mathbb{1}_{[t-1<\alpha\leq t]} \right] \\
+ E^\ast\left[ e^{-\int_0^T r(s)ds}G_{pp}(T)\mathbb{1}_{(T-1<T)} \right]. \\
\]
where
\begin{align}
I_1^{pp}(t) & := E^\pi \left[ e^{-\int_0^t r(s)ds} \max(e^{\psi(V(t))} - e^{-t}, 0) \right] \\
= & \frac{e^{-\alpha t}}{\pi} \int_0^\infty e^{-\psi(t, v)} (0, v) du, \\
I_2^{pp}(t) & := E^\pi \left[ e^{-\int_0^t r(s)ds} \max(e^{\psi(V(t))} - e^{-t}, 0) \right] \\
= & \frac{e^{-\gamma t}}{\pi} \int_0^\infty e^{-\gamma t} \psi(0, v) dv, \\
I_3^{pp}(t) & := E^\pi \left[ e^{-\int_0^t r(s)ds} e^{-t} \right] \\
= & e^{-\gamma t} \left[ \exp(A - \text{diag}(r')) t \right],
\end{align}

where \( r' := (r_1, r_2, \ldots, r_N, 0) \in \mathbb{R}^N \).

Proof. Again the proof is standard. The payoff in Eq. (14) can be decomposed as:
\[ G_{pp}(t) = \max(e^{\psi(V(t))} - e^{-t}, 0) - \max(e^{\psi(V(t))} - e^{-t}, 0) + e^{e^t}. \]

Since \( T(x) \) is independent of \( Y(t) \) and \( X(t) \), its distribution will not change when we change the real-world probability measure to the risk-neutral one (please also refer to Proposition 3.2 in Lin et al., 2009). Define \( \mathbb{E}_R \) as the indicator function of any event \( A \). Under the assumption that financial and mortality risks are independent, the time-zero value of the EIA contract is given by
\begin{align}
V_{pp}(0, T, \omega, \gamma, z) & = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(t) \right] 1_{\{t - 1 < T(x) \leq t\}} \right] \\
& \quad + E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(T) 1_{\{T(x) > T\}} \right] \\
& = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(t) \right] 2^\pi (t - 1 < T(x) \leq t) \right] \\
& \quad + E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(T) \right] 2^\pi (T(x) > T) \\
& = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(t) \right] \rho(t - 1 < T(x) \leq t) \right] \\
& \quad + E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(T) \right] \rho(T(x) > T) \\
& = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(t) \right] 1_{\{t - 1 < T(x) < t\}} \right] \\
& \quad + E^\pi \left[ e^{-\int_0^t r(s)ds} G_{pp}(T) \right] 1_{\{T(x) > T\}} \\
& = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} \max(e^{\psi(V(t))} - e^{-t}, 0) \right] \\
& \quad - \max(e^{\psi(V(t))} - e^{-t}, 0) + e^t \right] 1_{\{t - 1 < T(x) < t\}} \\
& \quad + E^\pi \left[ e^{-\int_0^t r(s)ds} \max(e^{\psi(V(t))} - e^{-t}, 0) \right] 1_{\{T(x) > T\}} \\
& = \sum_{i=1}^T \left[ I_1^{pp}(t) - I_2^{pp}(t) + I_3^{pp}(t) \right] 1_{\{t - 1 < T(x) < t\}} \\
& \quad + I_1^{pp}(T) - I_2^{pp}(T) + I_3^{pp}(T) 1_{\{T(x) > T\}}.
\end{align}

From Eqs. (21) and (22), \( I_1^{pp}(t) \) and \( I_2^{pp}(t) \) can be regarded as the time-zero values of time-\( t \) maturity call options with strike values \( e^t \) and \( e^{-t} \), respectively. Consequently, the pricing formulas for \( I_1^{pp}(t) \) and \( I_2^{pp}(t) \) can be derived by replacing \( k \) with \( e^t \) and \( e^{-t} \) directly as follows:
\begin{align}
I_1^{pp}(t) & = \frac{e^{-\alpha t}}{\pi} \int_0^\infty e^{-\psi(t, v)} \psi(0, v) dv, \\
I_2^{pp}(t) & = \frac{e^{-\gamma t}}{\pi} \int_0^\infty e^{-\gamma t} \psi(0, v) dv.
\end{align}

Furthermore, it is easy to see that
\begin{align}
I_1^{pp}(t) & = E^\pi \left[ e^{-\int_0^t r(s)ds} e^{-t} \right] \\
& = e^{-\alpha t} E^\pi \left[ e^{-\int_0^t r(s)ds} \right] \\
& = e^{-\alpha t} E^\pi \left[ e^{-\int_0^t r(s)ds} \right] 1_{\{X(0) \exp(A - \text{diag}(r')) t \}} \\
& = e^{-\gamma t} e^{-\gamma t} (X(0) \exp((A - \text{diag}(r')) t)),
\end{align}

where \( r' := (r_1, r_2, \ldots, r_N, 0) \in \mathbb{R}^N \). The last equality follows Buffington and Elliott (2002).

Typically, the implied critical participation rate, which is calculated by solving the equation letting the price be equal to 1, is generally reported in earlier works, see, for example, Lin et al. (2009). The following equation illustrates how to calculate these critical participation rates by solving the formula letting Eq. (20) be equal to 1, i.e.,
\[ V_{pp}(0, T, \omega, \gamma, z) = 1. \]

4.2. Valuation of the annual ratchet EIAs

In this subsection, the valuation of the annual ratchet EIAs is considered. The main feature of this kind of products is that the payoff functions are reset annually. In year \( t \), the contingent claims \( G_{pp}(t) \) are given by
\begin{align}
G_{pp}(t) & := \prod_{i=1}^t \max\{\min(e^{\omega_{ar}(i)} - Y(i) - Y(i - 1), \omega_{ar}), \omega_{ar} \},
\end{align}

where \( \omega_{ar} \) is the participation rate, \( z_{ar} \) is the minimum guarantee rate and \( \omega_{ar} \) is the cap rate for the annual ratchet EIA. Similarly, if the policyholder survives until the maturity of the contract, the policyholder will receive \( G_{ar}(T) \) at time \( T \). If the policyholder dies during the time period \( t, t + 1 \), for \( t = 1, 2, \ldots, T \), the beneficiary will receive \( G_{ar}(t) \) at \( t + 1 \). Consequently, the payoff of the annual ratchet EIA with mortality risk is given by
\[ G_{ar}(t), \quad t - 1 < T(x) \leq t, \quad \text{for } t = 1, 2, \ldots, T, \quad G_{ar}(T), \quad T(x) > T. \]

The following result is obvious, so we just state the result.

Proposition 4.2. Suppose an aged-x policyholder buys the \( T \)-maturity annual ratchet EIA contract. Then under the risk-neutral probability measure \( Q^\pi \), the time-zero value of the EIA contract \( V_{ar}(0, T, \omega_{ar}, \gamma_{ar}, z_{ar}) \) is given by
\begin{align}
V_{ar}(0, T, \omega_{ar}, \gamma_{ar}, z_{ar}) & = \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{ar}(t) \right] 1_{\{t - 1 < T(x) < t\}} \right] \\
& \quad + \sum_{i=1}^T \left[ E^\pi \left[ e^{-\int_0^t r(s)ds} G_{ar}(t) \right] 1_{\{T(x) > T\}} \right],
\end{align}

and the implied critical participation rate can be calculated by solving
\[ V_{ar}(0, T, \omega_{ar}^*, \gamma_{ar}, z_{ar}) = 1. \]
As pointed out in Lin et al. (2009), since it is impossible to know the state of the modulating Markov chain after the initiation time of the contract, it is difficult, if not impossible, to derive the analytical pricing formulas for the annual ratchet EIAs. Indeed, the return rates $Y(t) - Y(t - 1)$ within each year, for $t = 1, 2, \ldots, T$, are not independent under the regime-switching model. Instead, they are conditionally independent when $F^X(T)$ is given. So we cannot use the independence property to calculate the expectation of a discounted product payoff function in Eq. (26) as the product of the expectations of a series of independent processes. Consequently, the FFT method can be hardly applied to calculate the prices of the annual ratchet EIAs. In the numerical examples, we illustrate the valuation of the annual ratchet EIAs under a double regime-switching model by using Monte Carlo methods. In Lin et al. (2009), closed-form pricing formulas are obtained for some designs of EIAs in the case of a two-state modulating Markov chain and the absence of jumps in the asset price triggered by state transitions. Here we use a different approach based on the FFT to derive analytical pricing formulas for some designs of EIAs for an $N$-state modulating Markov chain and the presence of jumps in the asset price triggered by state transitions.

5. Valuation of variable annuities

In this section, we investigate the valuation of a variable annuity with GMDB under the double regime-switching model.

Note that the cost of GMDB in VAs should be considered in the valuation process. Here, we also suppose no management fee or other expenses are charged and let $\delta$ denote the continuously compounded guarantee charge. Then, the after-the-charge logarithmic return $\tilde{Y}(t)$ is governed by the following dynamics under the risk-neutral probability measure $\tilde{Q}$.

\[
d\tilde{Y}(t) := d\tilde{Y}(t) - \delta dt = \left[ r(t) - \delta - \frac{1}{2} \sigma^2(t) - \sum_{i=1}^{N} e^{\beta(t)\tilde{y}(t)} (e^{\tilde{y}(t)} - 1) - \beta(t) \tilde{y}(t) \right] dt + \sigma(t) d\tilde{W}^*(t) + \sum_{i=1}^{N} \beta_i(t) d\tilde{\Phi}^i(t), \quad t \in T. \tag{27}
\]

So the after-the-charge price of the investment fund satisfies $\tilde{S}(t) := e^{-\delta t} \tilde{S}(t)$. In year $t$, the payoff of the variable annuity $G_{DB}(t)$ is given by

\[
G_{DB}(t) := \max\{\tilde{S}(t), e^{\rho t}\},
\]

where $\rho$ is the continuously compounded minimum guarantee rate. Under this kind of contract, the larger one of the investment fund value and the death benefit will be paid to the beneficiary when the policyholder dies before the maturity time. If the policyholder survives until the contract expires, the value of the investment fund will be paid. Consequently, the payoff of a variable annuity with GMDB is given by

\[
G_{DB}(t), \quad t - 1 < T(x) \leq t, \quad \text{for } t = 1, 2, \ldots, T, \\
\tilde{S}(T), \quad T(x) > T.
\]

To simplify the notation, write $I_1^{DB}(t) := E^*\left[e^{-\rho t} \max(\tilde{Y}(t), 0)\right], I_2^{DB}(t) := E^*\left[e^{-\rho t} \tilde{y}(t) \tilde{y}(t)\right]$ and $I_3^{DB}(T) := E^*\left[e^{-\rho T} \tilde{S}(T)\right]$.

The following proposition is similar to Proposition 4.1.

**Proposition 5.1.** Suppose an aged-$x$ policyholder buys a $T$-maturity variable annuity with GMDB. Then the value of the GMDB in a variable annuity at time zero is given by

\[
V_{GMDB}(0, T, \delta, \rho) = \sum_{t=1}^{T} E^*\left[e^{-\rho t} \max(\tilde{Y}(t), 0)\right] + E^*\left[e^{-\rho T} \tilde{S}(T)\right] + \sum_{t=1}^{T} (I_1^{DB}(t) + I_2^{DB}(t) \cdot 1 + I_3^{DB}(T) \cdot \tilde{S}(T)), \tag{28}
\]

where

\[
i_1^{DB}(t) = E^*\left[e^{-\rho t} \max(\tilde{Y}(t), 0)\right] = \frac{1}{\pi} \int_{0}^{\infty} e^{-iv\tilde{Y}(t)} \psi^{DB}(0, t, v) dv,
\]

\[
i_2^{DB}(t) = e^{\rho t} e^{-\delta t} \tilde{S}(0) \exp((A^* - \text{diag}(\tilde{g}(r')) t) (1, 1),
\]

and

\[
i_3^{DB}(T) = E^*\left[e^{-\rho T} \tilde{S}(T)\right] = e^{-\delta T} \tilde{S}(0),
\]

with

\[
\psi^{DB}(0, t, v) = \exp(-i(v + (\alpha + 1)) \delta t) \times \frac{\tilde{S}(0) \exp((\text{diag}(g(v - i(\alpha + 1))) + B^*) t)}{\alpha^2 + \alpha - v^2 + (2\alpha + 1)i \alpha}, \tag{29}
\]

and $r' := (r_1, r_2, \ldots, r_N - 1, 0) \in \mathbb{R}^N$.

The proof is similar to Proposition 4.1, so we omit it.

6. Numerical examples

In this section, we perform numerical analysis for the valuation of the point-to-point EIAs, the annual ratchet EIAs and the GMDB in VAs under the double regime-switching model. As stated earlier, we adopt both the generalized regime-switching Esscher transform and the minimal martingale measure method to select equivalent martingale measures. Then, the FFT method is used to discretize the integral pricing formulas for the guarantees in the point-to-point EIAs and the GMDB in VAs. For the valuation of the annual ratchet EIA, the Monte-Carlo simulation is applied under the double regime-switching model.

To simplify our computation, we consider a two-state Markov chain $X$, where State 1 and State 2 of the chain represent a ‘Bad’ economy and a ‘Good’ one, respectively. Write $X(t) := (1, 0)$ and $X(t) := (0, 1)$ for State 1 and State 2, respectively. The classification of an economy here is based on the economic condition. For example, a “Good” economic state may represent the situation that the economy is booming, while a “Bad” economic state may represent the situation that the economy is in recession. In economics, different variables or indicators are used to measure the economic condition. Interest rate is a commonly used macroeconomic variable which describes the performance of an economy. When the economy is good (bad), interest rate is usually high (low). Furthermore, the volatility of a stock is generally low when the economy is good, while high when the economy performs badly. This may make sense from the economic point of view. Consequently, in our analysis, it is reasonable to define a ‘Bad’ economy as one with a low interest rate and a high volatility, while a ‘Good’ economy as one with a high interest rate and a low volatility. In practice, how to classify the state of an economy based on some economic indicators may be colored by some degrees of subjectivity. In particular, the choices of what economic indicators to be used or how to
draw a line to distinguish a “Good” economic state from a “Bad” one based on some economic indicators may depend on some subjective judgments of the users of the model. In what follows, we give configurations of the hypothetical parameters values. The rate matrix of the chain \( X \) under \( \mathcal{P} \) is given by

\[
A = \begin{pmatrix}
-a & a \\
a & -a
\end{pmatrix},
\]

where \( a \) takes discrete values from \([0, 0.1, 0.2, \ldots, 1]\) in our paper. Consider the following vectors for the appreciation rate, the risk-free interest rate, and volatility, respectively:

\[
\mu = (0.08, 0.10)', \quad r = (0.04, 0.08)', \quad \sigma = (0.3, 0.1)'.
\]

For ease of comparison, we also provide the numerical results for the valuation of the two kinds of equity-linked products under the single regime-switching model, which was the model considered in Fan (2013). Note that life table has been widely adopted to calculate the mortality of an insured by both academic researchers and market practitioners. Here we use the 2000–2003 China Life Table for males (see http://www.circ.gov.cn/) to depict the mortality rates of the policyholders in our model. The notifications of launching variable annuities announced by China Insurance Regulatory Commission (CIRC) in May 2011 means that variable annuities are introduced officially to Chinese insurance market. However, the development of these products are slow in the last three years. There are various reasons why the development of the variable annuities market in China is slow. Here we do not intend to focus on discussing these reasons. However, we believe that the development of scientific approaches to value variable annuities may have potentially constructive effects on the development of variable annuities market. So we use a China Life Table to illustrate how the model presented here may be applied to price variable annuities which may be related to those traded in the Chinese insurance market. For illustration, we use the life table for males since similar results can be derived based on the life table for females.

1. **The double regime-switching (DRS) model.**

The jump ratio of the double regime-switching model is determined by the following matrix

\[
\beta = \begin{pmatrix}
0 & \beta \\
-\beta & 0
\end{pmatrix}.
\]

2. **The single regime-switching (SRS) model.**

When the jump ratio remains zero during a state transition of the chain, the jump component of the double regime-switching model is absent. That is

\[
\beta = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

In other words, the double regime-switching model reduces to the single one.

6.1. **Numerical illustrations for the point-to-point EIA**

Table 1 presents the prices of point-to-point EIA with different cap rates \( \gamma \), different maturities \( T \) and different ages of the policyholders \( x \) under the DRS model and the SRS model with the pricing kernel selected by the regime-switching Esscher transform. Here, we assume \( \beta = 0.1 \) for the DRS model. For both models, suppose \( S_0 = 1, \alpha = 0.5, \omega = 0.8 \) and \( z = 0.03 \). In the table, the value of “Underestimation” refers to the percentages of underestimation of the EIA prices under the SRS model when compared with those under the DRS model. In our examples, we assume the number of discretization \( M = 4096 \) in the FFT approach. Note that the embedded option prices converge very quickly. Under the assumption of the above hypothetical parameter values, the regime-switching Esscher transform parameters for the ‘Bad’ and ‘Good’

<table>
<thead>
<tr>
<th>( x )</th>
<th>( T )</th>
<th>DRS model</th>
<th>State 1</th>
<th>State 2</th>
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states are \( \theta_1 = -0.4204 \) and \( \theta_2 = -1.3256 \), respectively. Then the risk-neutral intensities of the chain are given by \( \theta_{12} = 0.4794 \) and \( \theta_{21} = 0.5709 \). Furthermore, the market prices of the regime-switching risk can also be calculated as \( e^{\beta_1} - 1 = -0.0411 \) and \( e^{\beta_2} - 1 = 0.1417 \). We can see that regime switches have a significant impact on the market price of jump risk.

As shown in Table 1, the following results should be highlighted:

1. With the same minimum guarantee rate, the cap rate and the participation rate, the EIA prices in State 1 are systematically higher than those in State 2 under both models. These make intuitive sense. State 1 (‘Bad’ economy) has a lower interest rate and a higher volatility compared with State 2 (‘Good’ economy).

2. Consequently, it is reasonable that the EIA prices in State 1 are higher than the corresponding prices in State 2, which is partly due to the additional amount of risk premium required to compensate for a ‘Bad’ economic condition. Another possible explanation is that the lower interest rate and the higher volatility in State 1 could increase the probability that the guarantee is triggered, leading to a higher price in State 1.

3. Since the risk-free interest rate and the volatility of the investment fund value process, as well as the generator of the Markov chain are assumed to be the same under the DRS model and the SRS model, the DRS model apparently gives higher EIA prices due to additional jump risk induced by state transitions, (i.e. the EIA prices are underestimated under the SRS model). Although the underestimation of EIA prices are not considerably large, the jump risk should never be ignored.

4. It is worth noting that the underestimation percentages are higher in State 2 than those in State 1. When the age of the insured increases, the underestimation percentage decreases.

5. The prices of point-to-point EIA calculated with a cap rate are lower than those calculated without a cap rate \( (\gamma = \infty) \). The presence of a cap rate implies that the potential upside profit received by the policyholder is bounded. Consequently, it is not unreasonable that the prices of point-to-point EIA will be lower when there is a cap rate.

6. The prices of point-to-point EIA will increase when the policyholder’s age increases. One possible explanation is that the mortality risk of the policyholder increases when the
underwriting age of the policyholder increases. If the insurer faces a higher mortality risk, the prices of EIAs are more likely to increase. Another possible explanation is that the guarantee is more likely to be triggered by death of a more senior policyholder, which may lead to higher values of EIAs.

6. The prices of point-to-point EIAs are higher with shorter maturities. The explanation for this phenomenon could be similar with the one for item 5, since the effect of a higher policyholder’s age is similar to that of a shorter maturity.

Table 2 presents the prices of point-to-point EIAs with different cap rates $\gamma$, different maturities $T$ and different ages of the policyholders $x$ under the DRS model and the SRS model with the minimal martingale measure. To compare the EIAs prices based on these two pricing kernels, we adopt the same configurations of parameter values as those adopted in Table 1. The values of “Underestimation” are percentages of underestimation of the EIA prices under the SRS model when compared with those under the DRS model. Comparing the results in Tables 1 and 2, the prices of point-to-point EIAs calculated under the SRS model with the two pricing kernels are exactly the same. This is in line with the explanations in Remark 3.3. The prices of point-to-point EIAs under the DRS model obtained by the two equivalent martingale measures are very close. In what follows, unless otherwise stated, we always apply the Esscher transformed equivalent martingale measure to calculate the prices of the annuity guarantees.

Table 3 presents the critical participation rates for the point-to-point EIA under the DRS model and the SRS model, respectively. Here, we assume $\beta = 0.1$ for the DRS model and $S_0 = 1$, $T = 10$, $\alpha = 0.5$ and $z = 0.03$ for both models. In the table, the numbers in parentheses under the critical participation rates represent the percentages of underestimation of the critical participation rates under the SRS model when compared with those under the DRS model. The results for the aged-55, aged-60 and aged-65 policyholders are provided as well. From the comparisons between Tables 1 and 3, we notice that the results of the EIA prices and the critical participation rates display totally opposite trends. These make intuitive sense. The following simple example is to illustrate the idea. There are two insurers, Insurer A and Insurer B, selling EIAs. Insurer A, may experience a higher risk than Insurer B. The two insurers have the same assumptions except the participation rate and the price of the EIA. If the two insurers are required to have the same participation rates, Insurer A will definitely increase the prices of EIAs to manage the higher risk. If the prices of EIAs provided by the two insurers are required to be the same, Insurer A can only decrease the participation rate to limit the potential upside profit the customers could gain. So it is not unreasonable that the results in Tables 1 and 3 show opposite trend. Consequently, the explanations for Table 3 can be similarly presented as those for Table 1.

Furthermore, we also assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$, $\alpha = 0.5$ and $\gamma = 0.2$ to illustrate EIA prices under the DRS model with different levels of participate rate $\omega$ in both State 1 and State 2. From Figs. 1 and 2, the EIA prices increase with the value of the participation rate $\omega$ in both states. In addition to the explanations earlier, Figs. 1 and 2 also illustrate the interplay between the EIA price and the participation rate. When other assumptions remain the same, a higher participation rate implies that the policyholder could gain higher potential upside profit, leading to a higher price of the EIA contract.

As in Table 1, we have discussed the EIA prices obtained under the DRS model with cap rate and without cap rate, respectively. Here, we provide the sensitivity analysis for the EIA prices with respect to the cap rate $\gamma$. Under the DRS model, we assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$, $\alpha = 0.5$ and $\omega = 0.8$. Figs. 3 and 4 illustrate that the prices of point-to-point EIAs will increase when the cap rate $\gamma$ increases. This may be attributed to the higher potential upside profit the policyholder may gain. A larger cap
Fig. 2. EIA prices corresponding to different $\omega$ with $z = 0.04$.

Fig. 3. EIA prices corresponding to different $\gamma$ with $z = 0.03$.

Fig. 4. EIA prices corresponding to different $\gamma$ with $z = 0.04$.

Fig. 5. EIA prices corresponding to different $a$ with $z = 0.03$.

The rate implies that the policyholder could receive a higher potential upside profit, leading to a higher value of the EIA contract.

Under the DRS model, we assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$, $\omega = 0.8$ and $\gamma = 0.2$. A sensitivity analysis for the EIA prices with respect to the rate of transition $a$ is presented. Note that the probability of the transitions between different states of the economy increases with $a$, which implies a more volatile economy. The regime-switching effect is degenerate when $a = 0$. From Figs. 5 and 6, we notice that the prices of the EIAs decrease with $a$ in State 1 while increase with $a$ in State 2. As explained earlier, the EIAs are more expensive in State 1 (bad state) than in State 2 (good state). From this aspect, the prices of EIAs should decrease in State 1. However, the higher transition intensity in State 1 will lead to a higher transition probability of the chain from State 1 to State 2, hence a higher frequency of triggered jumps. Seen from this aspect, the market price of the jump risk may increase when the transition intensity $a$ increases. Consequently, we further calculate the market prices of the diffusion risk and the jump risk for different transition intensities $a$. The market prices of the diffusion risk are $0.4444, 0.4345, 0.4250, 0.4159, 0.4071, 0.3988$ when the values of $a$ are $0, 0.2, 0.4, 0.6, 0.8, 1$ respectively. Meanwhile, the market prices of the jump risk are $-0.0435, -0.0425, -0.0416, -0.0407, -0.0399, -0.0391$, respectively. The negative market prices of the jump risk from State 1 to State 2 imply that the market will compensate investors bearing the jump risk from State 1 to State 2. This is because an upward jump in the price level of the underlying investment fund will occur when the economy switches from a 'Bad' state into a 'Good' state. Under our configurations of the hypothetical values of the model parameters, the market prices of the diffusion risk decrease with $a$, while those of the jump risk increase with $a$. Consequently, the trend of the EIA prices with increasing transition intensities is determined by the combined effects of both the diffusion risk and the jump risk. Note that the market prices of the diffusion risk are approximately ten times of those of the jump risk in absolute values. Seen from such numerical results, although the jump may occur more frequently when $a$ increases, the market prices of the diffusion risk is the dominant factor. This provides some evidence that why the EIA prices will decrease when the transition intensity increases in State 1. On the other hand, the EIA prices in State 2 will increase when $a$ increases. This makes intuitive sense. When $a = 0$, the transition probabilities of the chain between State 1 and State 2 are zero, implying the regime switching effect does not exist under this degenerate case. Consequently, the EIA prices are the maximal in State 1 and the minimal in State 2 when $a = 0$.

Furthermore, from Figs. 1–6, it is worth noting that the EIA prices calculated with $z = 0.04$ are higher than those calculated with $z = 0.03$. This also makes intuitive sense since a higher cost of guarantee, implied by a larger minimum guarantee, will lead to a higher price.
The parameter configurations are assumed to be the same as the ratchet EIAs under the DRS model and the SRS model, respectively. EIAs are considerably lower than those for the point-to-point EIAs, with Table 3, the critical participation rates for the annual ratchet as those for the point-to-point EIA valuation. Comparing Table 4 of the results for the annual ratchet EIA valuation can be given as those for the point-to-point EIA valuation. Comparing Table 4 with Table 3, the critical participation rates for the annual ratchet EIAs are considerably lower than those for the point-to-point EIAs, which implies that the ratchet EIAs are much more expensive than the point-to-point EIAs. In addition, the overestimating percentage of the annual ratchet EIA participation rates under the SRS model when compared with those under the DRS model are much higher than those of the point-to-point EIA participation rates.

6.3. Numerical illustrations for the GMDB

Tables 5 and 6 present the prices of the GMDB in variable annuities with different minimum guarantee rates, different ages and different maturities under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure and the minimal martingale measure, respectively. Here, we assume $\beta = 0.1$ for the DRS model. For both models, suppose $S_0 = 1$, $a = 0.5$, and $\delta = 0.03$. In Tables 5 and 6, the numbers of “Underestimation” represent the percentages of underestimation of the GMDB prices under the SRS model when compared with those under the DRS model.

From Tables 5 and 6, we have the following observations.

1. With the same minimum guarantee rate and maturity time, the prices of the GMDB in VAs in State 1 are systematically higher than those in State 2 under both models. This makes economic sense since State 1 (State 2) is a ‘Bad’ (‘Good’) one.
2. The prices of the GMDB in VAs are higher under the DRS model than those under the SRS model with the same configuration of hypothetical parameters values.
3. When the minimum guarantee rate increases, the prices of the GMDB in VAs increase as well. This makes intuitive sense. To provide a higher minimum guarantee rate, the insurers pay more attention to make investment strategies, implying a higher cost. Consequently, the prices of GMDB in VAs are higher than those with lower minimum guarantee rates.
4. The prices of the GMDB in VAs increase when the policyholder’s age increases.
5. Contracts of the GMDB in VAs are more expensive when the maturity time is shorter.

Note that possible explanations for the above observations are similar as those for the valuation of the point-to-point EIAs. It is worth mentioning that Tables 1 and 5 show that the EIA prices and the

### Table 4

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<th>SRS model</th>
<th>Underestimation</th>
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<td>State 2</td>
<td>State 1</td>
</tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(−9.09%)</td>
</tr>
<tr>
<td>60</td>
<td>0.3787</td>
<td>0.4335</td>
<td>0.4106</td>
</tr>
<tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(−8.42%)</td>
</tr>
<tr>
<td>65</td>
<td>0.3774</td>
<td>0.4295</td>
<td>0.4072</td>
</tr>
<tr>
<td></td>
<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(−7.90%)</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>$x$</th>
<th>$T$</th>
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<th>SRS model</th>
<th>Underestimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>State 1</td>
<td>State 2</td>
<td>State 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td>6</td>
<td>0.8476</td>
<td>0.8459</td>
<td>0.8469</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.8086</td>
<td>0.8064</td>
<td>0.8074</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.7760</td>
<td>0.7731</td>
<td>0.7742</td>
</tr>
<tr>
<td>60</td>
<td>6</td>
<td>0.8570</td>
<td>0.8540</td>
<td>0.8558</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.8244</td>
<td>0.8205</td>
<td>0.8224</td>
</tr>
<tr>
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<td>0.8001</td>
<td>0.7951</td>
<td>0.7970</td>
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<tr>
<td>65</td>
<td>6</td>
<td>0.8714</td>
<td>0.8664</td>
<td>0.8694</td>
</tr>
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<td>0.8486</td>
<td>0.8420</td>
<td>0.8453</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.8364</td>
<td>0.8284</td>
<td>0.8315</td>
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Table 6
Prices of the GMDB in VAs under the DRS model and the SRS model with the minimal martingale measure.

<table>
<thead>
<tr>
<th>x</th>
<th>T</th>
<th>DRS model</th>
<th>SRS model</th>
<th>Underestimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>State 1</td>
<td>State 2</td>
<td>State 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.8476</td>
<td>0.8458</td>
<td>0.8469</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.08%</td>
<td>0.11%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>55</td>
<td>0.8086</td>
<td>0.8062</td>
<td>0.8074</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.15%</td>
<td>0.16%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.7759</td>
<td>0.7729</td>
<td>0.7742</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.22%</td>
<td>0.26%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.8569</td>
<td>0.8539</td>
<td>0.8558</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.13%</td>
<td>0.16%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.8243</td>
<td>0.8203</td>
<td>0.8224</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.23%</td>
<td>0.28%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.7999</td>
<td>0.7948</td>
<td>0.7970</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.36%</td>
<td>0.42%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.8713</td>
<td>0.8662</td>
<td>0.8694</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.22%</td>
<td>0.28%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.8484</td>
<td>0.8417</td>
<td>0.8453</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.37%</td>
<td>0.45%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.8361</td>
<td>0.8278</td>
<td>0.8315</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.55%</td>
<td>0.64%</td>
<td></td>
</tr>
</tbody>
</table>

Table 7
Fair guarantee charge for the GMDB in VAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure.

<table>
<thead>
<tr>
<th>x</th>
<th>DRS model</th>
<th>SRS model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State 1</td>
<td>State 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>55</td>
<td>0.140</td>
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<td>0.14%</td>
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<tr>
<td></td>
<td>60</td>
<td>0.250</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.30%</td>
</tr>
<tr>
<td></td>
<td>65</td>
<td>0.450</td>
</tr>
</tbody>
</table>

GMDB prices under the SRS model are lower than those under the DRS model. Consequently compared to those prices in Lin et al. (2009), the prices obtained from the DRS model provide more conservative estimates for the GMDB prices to compensate for the additional jump risk triggered by state transitions.

With the same configurations of the parameter values, Table 7 presents the fair guarantee charges for the GMDB in VAs under the DRS model and the SRS model corresponding to the GMDB prices in Table 5. As indicated in Table 7, the fair guarantee charges in State 1 are systematically higher than those in State 2 under both models. These make intuitive sense. State 1 ('Bad' economy) has a lower interest rate and higher volatility compared with State 2 ('Good' economy). Consequently, the fair guarantee charges in State 1 will be higher than those in State 2 due to the higher cost to maintain the minimum guarantee. This is also one possible explanation for the results that the DRS model requires higher fair guarantee charge than the SRS model.

Under the DRS model, we assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$ and $\delta = 0.5$. Figs. 7 and 8 present the sensitivity analysis for the GMDB prices with respect to $\delta$. A larger $\delta$ implies a higher cost to maintain the minimum guarantee, leading to a lower GMDB price. This gives an intuitive description of the relationship between the GMDB prices and the fair guarantee charges.

Under the assumptions that $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$ and $\delta = 0.03$, we also illustrate GMDB prices under the DRS model with different levels of $a$ in both State 1 and State 2. Seen from Figs. 9 and 10, the GMDB prices decrease with $a$ in State 1, while increase with $a$ in State 2. The explanations are the same with those for the valuation of the point-to-point EIAs.
GMDB prices with minimum guarantee rate $\gamma = 0.03$

7. Conclusions

We considered the valuation of the point-to-point EIAs, the annual ratchet EIAs and the GMDB in VAs under the double regime-switching model, which can endogenously determine the regime-switching risk. To select an equivalent martingale measure, we applied the generalized version of the regime-switching Esscher transform and the minimal martingale measure method. This can quantify the regime-switching risk. Then under the risk-neutral probability measure selected by either the Esscher transformed equivalent martingale measure or the minimal martingale measure, analytical pricing formulas for the two products were obtained using the inverse Fourier transform method. The FFT method is adopted to discretize the analytical pricing formulas. Numerical examples illustrated the regime-switching effects on the prices of the point-to-point EIAs, the annual ratchet EIAs and GMDB in VAs. Furthermore, our numerical examples could illustrate the impacts of different models and parameters on the prices of the equity-linked products. As given in Tables 1 and 5, the SRS model adopted in Lin et al. (2009) might underestimate the prices of equity-linked products. By incorporating those jumps triggered by state transitions, we can endogenously determine the regime-switching risk. The sensitivity analysis of different parameters also provides some intuition about the qualitative price comparisons.

Since the terms of the EIAs and other VA contracts are as long as several decades, it is natural to consider the uncertainty of mortality rate in such a long period. Consequently, it may be interesting to incorporate stochastic mortality in the current modeling framework. Another potential research direction is the derivation of analytical pricing formulas for VA contracts with some complicated optionality structures under the double regime-switching model. It seems that the Laplace transform may provide some clues to this possibly challenging problem.

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References


